Painlevé equations from Nakajima-Yoshioka blow-up relations

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Isomonodromy/CFT correspondence

The Painlevé VI equation is a particular case of the equation of the isomonodromic deformation of linear differential equation.

Painlevé VI tau function

$$\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}_{c=1}(\vec{\theta}, \sigma + n|z). \quad (1)$$

- $\mathcal{Z}_{c=1}(\vec{\theta}, \sigma + n|z)$ — Virasoro conformal block with $c = 1$.
- By AGT $\mathcal{Z}_{c=1}$ — 4d Nekrasov partition function $SU(2)$ with $\epsilon_1 + \epsilon_2 = 0$
- irregular singularities — irregular conformal blocks — another number of matter fields
- isomonodromic deformation of rank N linear system — W_N conformal blocks with $c = N - 1$ — 4d Nekrasov partition function $SU(N)$ with $\epsilon_1 + \epsilon_2 = 0$.

Incomplete list of people: [Gamayun, Iorgov, Lisovyy, Teschner, Shchechkin, Gavrylenko, Marshakov, Its, Bonelli, Grassi, Tanzini, Nagoya, Tykhyy, Maruyoshi, Sciarappa, Mironov, Morozov, Iwaki, Del Monte,. . .]

Another central charges

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There are several reasons to believe the existence of such analogue for central charges of (logarithmic extension of) minimal models $\mathcal{M}(1, n)$

$$c = 1 - 6\frac{(n-1)^2}{n}, n \in \mathbb{Z} \setminus \{0\}. \quad (2)$$

- Operator valued monodromies commute [Iorgov, Lisovyy, Teschner 2014].
- Bilinear relations on conformal blocks [MB., Shchechkin 2014]
- Action of $SL(2, \mathbb{C})$ on the vertex algebra [Feigin 2017]

Today: $c = -2$ tau functions

$$\tau^{\pm}(a, s|z) = \sum_{n \in \mathbb{Z}} s^{n/2} \mathcal{Z}(a + 2n\epsilon; \mp\epsilon, \pm 2\epsilon|z). \quad (3)$$

Blow-up relations

$$\tau(a,s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(a+2n\epsilon,\epsilon,-\epsilon|z), \tag{4}$$

$$\tau^{\pm}(a,s|z) = \sum_{n\in\mathbb{Z}} s^{n/2} \mathcal{Z}(a+2n\epsilon;\mp\epsilon,\pm 2\epsilon|z). \tag{5}$$

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- [Nakajima Yoshioka 03, 05, 09], [Göttshe, Nakajima, Yoshioka 06], [MB, Feigin, Litvinov 13],

  $$\beta_D \mathcal{Z}(a,\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}+j/2} \mathrm{D}\Big(\mathcal{Z}(a+n\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2|z), \mathcal{Z}(a+n\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2|z)\Big),$$

  D is some differential operator, $j=0,1$, β_D is some function (may be zero).
- Now set $\epsilon_1+\epsilon_2=0$, and take the sum of these relations with coefficients s^n

$$\beta_D \tau(z) = \mathrm{D}(\tau^+(z),\tau^-(z)). \tag{6}$$

  Excluding $\tau(z)$ one gets system of bilinear relations on $\tau^+(z)$, $\tau^-(z)$.
- This system can be used to prove the (Painlevé) bilinear relations on $\tau(z)$

Blow-up relations for $\mathbb{C}^2/\mathbb{Z}_2$

$$\tau(a, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(a + 2n\epsilon, \epsilon, -\epsilon|z), \tag{7}$$

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- [Bruzzo, Poghossian, Tanzini 09], [Bruzzo, Pedrini, Sala, Szabo 2013], [Ohkawa 2018], [Belavin, MB., Feigin, Litvinov, Tarnopolsky 2011]

$$\tilde{\mathcal{Z}}(a,\epsilon_1,\epsilon_2|z) = \sum_n \mathrm{D}\Big(\mathcal{Z}(a+n\epsilon_1, 2\epsilon_1, -\epsilon_1+\epsilon_2|z), \mathcal{Z}(a+n\epsilon_2, \epsilon_1-\epsilon_2, 2\epsilon_2|z)\Big). \tag{8}$$

Here $\tilde{\mathcal{Z}}$ is Nekrasov partition function for $\mathbb{C}^2/\mathbb{Z}_2$.

- After specialization $\epsilon_1+\epsilon_2=0$ and exclusion $\tilde{\mathcal{Z}}$ we get (Painlevé) bilinear relations on $\tau(z)$ [MB., Shchechkin 2014]

$$\tilde{\mathrm{D}}(\tau(z),\tau(z)) = 0. \tag{9}$$

So we derive (some) $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equation from ordinary $\mathbb{C}^2$ blow-up equations (in case $\epsilon_1+\epsilon_2=0$).

Plan of the talk

1. Introduction
2. Example: parameterless Painlevé equation
3. Example: parameterless q-difference Painlevé equation
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Painlevé III($D_8^{(1)}$) equation

Another name for this equations Painlevé III_3'.

- Toda-like form of these equation is a two bilinear equations on two functions: $\tau = \tau_0$ and τ_1. It is symmetric under $\tau_0 \leftrightarrow \tau_1$.
- Painlevé III_3

$$\begin{aligned} D^2_{[\log z]}(\tau_0, \tau_0) &= -2z^{1/2}\tau_1^2 \\ D^2_{[\log z]}(\tau_1, \tau_1) &= -2z^{1/2}\tau_0^2 \end{aligned} \tag{10}$$

where second Hirota differential $D^2_{[\log z]}(\tau, \tau) = 2\tau''\tau - \tau'^2$, $f' = z\frac{df}{dz}$.

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- Solution

$$\tau_j(a, s|z) = \sum_{n \in \mathbb{Z}+j/2} s^n \mathcal{Z}(a + 2n\epsilon, \epsilon, -\epsilon|z), \quad j = 0, 1. \qquad (11)$$

  Here $\mathcal{Z}(a, \epsilon_1, \epsilon_2|z)$ — Nekrasov function for 4d pure SU(2) gauge theory. Here a, s are integration constants for Painlevé equation.
- In CFT notations $c = 1 + 6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$, $\sigma = -\frac{a}{2\epsilon_1}$
- The equations (10) could be rewritten as single equation on $\tau(a, s|z)$

$$D^2_{[\log z]}(\tau(\sigma, s|z), \tau(\sigma, s|z)) = -2z^{1/2}\tau(\sigma + 1/2, s|z)\tau(\sigma - 1/2, s|z). \qquad (12)$$

Blow-up relations

- They express instanton partition function on $\widehat{\mathbb{C}^2} = (\mathbb{C}^2$ blowed-up in the point) as a bilinear expression on $\mathbb{C}^2$ instanton partition function

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|\Lambda) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_1 n|\epsilon_1,\epsilon_2-\epsilon_1|\Lambda)\mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_2 n|\epsilon_1-\epsilon_2,\epsilon_2|\Lambda) \quad (13)$$

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|\Lambda) = \mathcal{Z}_{\mathbb{C}^2}(a|\epsilon_1,\epsilon_2|\Lambda) \quad (14)$$

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$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|\Lambda) = \mathcal{Z}_{\mathbb{C}^2}(a|\epsilon_1,\epsilon_2|\Lambda) \quad (14)$$

- Imposing condition $\epsilon_1+\epsilon_2=0$ we get in the CFT notations

$$\mathcal{Z}_{c=1}(\sigma|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}^{+}_{c=-2}\left(\sigma-n\,\middle|\,\frac{z}{4}\right)\mathcal{Z}^{-}_{c=-2}\left(\sigma+n\,\middle|\,\frac{z}{4}\right), \quad (15)$$

We get $\tau(\sigma,s|z) = \tau^{+}(\sigma,s|z)\tau^{-}(\sigma,s|z)$,

Recall that in CFT notation

$$\tau(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}_{c=1}(\sigma+n|z), \quad \tau^{\pm}(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}\mathcal{Z}^{\pm}_{c=-2}(\sigma+n|z/4).$$

Blow-up relations 2

We get $\qquad \tau(\sigma,s|z)=\tau^{+}(\sigma,s|z)\tau^{-}(\sigma,s|z),$

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Differential blow-up relations

$$\sum_{n\in\mathbb{Z}} \mathcal{Z}(a+2\epsilon_1 n; \epsilon_1, \epsilon_2-\epsilon_1|\Lambda e^{-\frac{1}{2}\epsilon_1\alpha})\mathcal{Z}(a+2\epsilon_2 n; \epsilon_1-\epsilon_2, \epsilon_2|\Lambda e^{-\frac{1}{2}\epsilon_2\alpha})|_{\alpha^4} =$$
$$= \frac{(2\alpha)^4}{4!}\left(\left(\frac{\epsilon_1+\epsilon_2}{4}\right)^4 - 2\Lambda^4\right)\mathcal{Z}(a;\epsilon_1,\epsilon_2|\Lambda) + O(\alpha^5). \tag{16}$$

We get

$$D^1_{[\log z]}(\tau^+, \tau^-) = z^{1/4}\tau_1, \qquad D^2_{[\log z]}(\tau^+, \tau^-) = 0,$$
$$D^3_{[\log z]}(\tau^+, \tau^-) = z^{1/4}\left(z\frac{d}{dz}\right)\tau_1, \qquad D^4_{[\log z]}(\tau^+, \tau^-) = -2z\tau. \tag{17}$$

Painlevé equations from Nakajima-Yoshioka blow-up relations

$$\tau_0 = \tau^+\tau^-,\ D^1_{[\log z]}(\tau^+,\tau^-) = z^{1/4}\tau_1,\ D^2_{[\log z]}(\tau^+,\tau^-) = 0. \quad (18)$$

Theorem

Let $\tau^\pm$ satisfy equations (18). Then τ_0 and τ_1 satisfy Toda-like equation

$$D^2_{[\log z]}(\tau_0,\tau_0) = -2z^{1/2}\tau_1^2 \quad (19)$$

Since we know from blow-up relations that $\tau^\pm(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}\mathcal{Z}^\pm_{c=-2}(\sigma+n|z/4)$ *satisfy* (18) *we proved that* $\tau(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^n\mathcal{Z}_{c=1}(\sigma+n|z)$ *satisfy Painlevé equation.*

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Difference equations

Painlevé $A_7^{(1)'}$ equation.

- Toda-like form of these equation is a two bilinear equations on two functions: $\tau = \tau_0$ and τ_1. It is symmetric under $\tau_0 \leftrightarrow \tau_1$.

$$\begin{aligned} \overline{\tau_0}\underline{\tau_0} &= \tau_0^2 - z^{1/2}\tau_1^2 \\ \overline{\tau_1}\underline{\tau_1} &= \tau_1^2 - z^{1/2}\tau_0^2 \end{aligned} \tag{20}$$

where $\overline{\tau(z)} = \tau(qz), \underline{\tau(z)} = \tau(q^{-1}z)$.

- Solution

$$\tau_j(a, s|z) = \sum_{n\in\mathbb{Z}+j/2} s^n \mathcal{Z}(a + 2n\epsilon, \epsilon, -\epsilon|z),\ j = 0, 1. \tag{21}$$

Here $\mathcal{Z}(a, \epsilon_1, \epsilon_2|z)$ — Nekrasov function for pure 5d SU(2) gauge theory. Here a, s are integration constants for Painlevé equation. $q = e^{R\epsilon}$, $u = e^{Ra}$.

- The equations (20) could be rewritten as single equation on $\tau(u, s|z)$

$$\tau(u, s|qz)\tau(u, s|q^{-1}z) = \tau^2(u, s|z) - z^{1/2}\tau(uq, s|z)\tau(uq^{-1}, s|z). \tag{22}$$

Blow-up relations

$$\begin{aligned} \mathrm{\tau}^+\mathrm{\tau}^- &= \tau \\ \overline{\mathrm{\tau}^+}\,\underline{\mathrm{\tau}^-} - \underline{\mathrm{\tau}^+}\,\overline{\mathrm{\tau}^-} &= -2z^{1/4}\tau_1,. \\ \overline{\mathrm{\tau}^+}\,\underline{\mathrm{\tau}^-} + \underline{\mathrm{\tau}^+}\,\overline{\mathrm{\tau}^-} &= 2\tau \end{aligned} \tag{23}$$

Theorem

Take (23), then τ and τ_1 satisfy Toda-like equation

$$\overline{\tau}\,\underline{\tau} = \tau^2 - z^{1/2}\tau_1^2. \tag{24}$$

Proof: $$\overline{\mathrm{\tau}^+\mathrm{\tau}^-}\,\underline{\mathrm{\tau}^+\mathrm{\tau}^-} = \frac{1}{4}(\overline{\mathrm{\tau}^+}\,\underline{\mathrm{\tau}^-} + \underline{\mathrm{\tau}^+}\,\overline{\mathrm{\tau}^-})^2 - \frac{1}{4}(\overline{\mathrm{\tau}^+}\,\underline{\mathrm{\tau}^-} - \underline{\mathrm{\tau}^+}\,\overline{\mathrm{\tau}^-})^2 \tag{25}$$

Since we know from blow-up relations that $\mathrm{\tau}^\pm(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}\mathcal{Z}^\pm_{c=-2}(\sigma+n|z/4)$ *satisfy* (23) *we proved that* $\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n\mathcal{Z}_{c=1}(\sigma+n|z)$ *satisfy q -Painlevé equation.*

For another proof see [Matsuhira, Nagoya 2018].

Chern-Simons generalization

$$\tau(u,s|qz)\tau(u,s|q^{-1}z) = \tau^2(u,s|z) - z^{1/2}\tau(uq,s|z)\tau(uq^{-1},s|z). \qquad (26)$$

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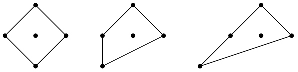
- In the work [MB, Marshakov, Gavrylenko 2018] there was considered generalization of the Toda-like equation (26). This generalization depends on two integer parameters $N \in \mathbb{N}, 0 \le m \le N$ and has the form

$$\tau_{m;j}(qz)\tau_{m;j}(q^{-1}z) = \tau_{m;j}(z)^2 - z^{1/N}\tau_{m;j+1}(q^{m/N}z)\tau_{m;j-1}(q^{-m/N}z),\ j \in \mathbb{Z}/N\mathbb{Z}.$$

- Here $N=2$. The solutions are given by

$$\tau_{m,j}(u,s|z) = \sum_{n\in\mathbb{Z}+j/2} s^n \mathcal{Z}_m(uq^{2n}|z). \qquad (27)$$

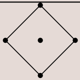
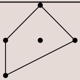
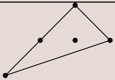
  where $\mathcal{Z}_m$ is a 5d Nekrasov function for $SU(N)$ with Chern-Simons level m.
- Newton polygons:



Theorem

Formula (27) follows from blow-up relations for $N=2$.

Chern-Simons generalization: $m=0$ vs. $m=2$

Chern-Simons level	$m=0$	$m=1$	$m=2$
Newton polygon			
Painlevé equation	*q-Painlevé* $A_7^{(1)'}$	*q-Painlevé* $A_7^{(1)}$	*q-Painlevé* $A_7^{(1)'}$

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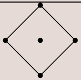
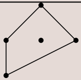
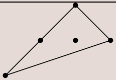
Newton polygon			
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- for $m = 0$ and $m = 2$ Painlevé equations are the same
- We have relation on the level of tau functions

$$\tau_j = (qz; q, q)_\infty \tau_{2;j} \tag{28}$$

- We have relations on the Nekrasov functions

$$\mathcal{Z}_2(u; q^{-2}, q|z) = (z; q^{-2}, q)_\infty \mathcal{Z}_0(u; q^{-2}, q|z), \tag{29}$$
$$\mathcal{Z}_2(u; q^{-1}, q^2|z) = (z; q^{-1}, q^2)_\infty \mathcal{Z}_0(u; q^{-1}, q^2|z), \tag{30}$$
$$\mathcal{Z}_2(u; q^{-1}, q|z) = (z; q^{-1}, q)_\infty \mathcal{Z}_0(u; q^{-1}, q|z). \tag{31}$$

We prove this from blow-up relations. (Another reference ?)

Connection with ABJ theory

- [Bonelli Grassi Tanzini 17] proposed
$$\tau_{\mathrm{BGT}}(u|z) = \sum_{n\in\mathbb{Z}+j/2} \mathcal{Z}(uq^{2n}, q, q^{-1}|z). \qquad (32)$$
Here $|q|=1$, the function $\mathcal{Z}$ is redefined by adding certain (non-perturbative) corrections, $s=1$.
- By the topological string/spectral theory duality [Grassi Hatsuda Marino 2014] the function τ_{BGT} essentially equals to a spectral determinant of an operator
$$\rho = (e^{\hat{p}} + e^{-\hat{p}} + e^{\hat{x}} + me^{-\hat{x}})^{-1}. \qquad (33)$$
Here operators $\hat{x}, \hat{p}$ satisfy commutation relation $[\hat{x},\hat{p}] = i\hbar$.
Parameters related by $\hbar = \frac{4\pi^2 i}{\log q}$, $m = \exp\left(\frac{-\hbar\log z}{2\pi}\right)$.
- Denote by $\Xi(\kappa, z) = \det(1+\kappa\rho)$ a spectral (Fredholm) determinant of the ρ.

$$\tau_{\mathrm{BGT}}(u|z) = Z_{\mathrm{CS}}(z)\Xi(\kappa, z). \qquad (34)$$

The auxiliary function Z_{CS} is given in by an explicit expression and satisfy
$$\overline{Z_{\mathrm{CS}}(z)}\underline{Z_{\mathrm{CS}}(z)} = (z^{1/4} + z^{-1/4})Z^2_{\mathrm{CS}}(z). \qquad (35)$$

Connection with ABJ theory: Wronskian-like relations

- In the special case $z = q^M$, $M \in \mathbb{Z}$ the spectral determinant of the operator ρ simplifies and equals to the grand canonical partition function of the ABJ theory.
- $\Xi(\kappa, z)$ can be factorised according to the parity of the eigenvalues of ρ

$$\Xi(\kappa, z) = \Xi^+(\kappa, z)\Xi^-(\kappa, z). \tag{36}$$

It was conjectured in [Grassi Hatsuda Marino 2014] that functions Ξ^+, Ξ^- satisfy additional (Wronskian-like) relations

$$\begin{aligned} iz^{1/4}\overline{\Xi_1^+}\,\underline{\Xi_1^-} - \overline{\Xi^+}\,\underline{\Xi^-} &= (iz^{1/4} - 1)\Xi^+\Xi^-, \\ iz^{1/4}\underline{\Xi_1^+}\,\overline{\Xi_1^-} + \underline{\Xi^+}\,\overline{\Xi^-} &= (iz^{1/4} + 1)\Xi^+\Xi^-. \end{aligned} \tag{37}$$

Here Ξ_1 is Bäcklund transformation of the Ξ, in terms of κ it is $\kappa \to -\kappa$.

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Here Ξ_1 is Bäcklund transformation of the Ξ, in terms of κ it is $\kappa \to -\kappa$.

Theorem (/Conjecture)

The equations (37) are equivalent to the blow-up relations, where $\Xi^\pm = Z_{\mathrm{CS}}^\pm \tau^\pm$.

- Here $\overline{Z_{\mathrm{CS}}^+}\underline{Z_{\mathrm{CS}}^-} = (1 + iz^{1/4})Z_{\mathrm{CS}}^+ Z_{\mathrm{CS}}^-, \quad \underline{Z_{\mathrm{CS}}^+}\overline{Z_{\mathrm{CS}}^-} = (1 - iz^{1/4})Z_{\mathrm{CS}}^+ Z_{\mathrm{CS}}^-$
- Topological string/spectral theory duality for the case $t = q^2$?

Discussion

- For Painlevé VI the simplest of the Nakajima-Yoshioka relations leads to

$$\tau(\overrightarrow{\theta};\sigma,s|z)=\tau(\overrightarrow{\theta}+\frac{1}{2}e_{23};\sigma,s|z)\tau(\overrightarrow{\theta}-\frac{1}{2}e_{23};\sigma,s|z)$$
$$+\tau(\overrightarrow{\theta}+\frac{1}{2}e_{23};\sigma+1,s|z)\tau(\overrightarrow{\theta}-\frac{1}{2}e_{23};\sigma-1,s|z), \quad (38)$$

  where $\overrightarrow{\theta}=(\theta_0,\theta_t,\theta_1,\theta_\infty)$, $e_{23}=(0,1,1,0)$ and τ is the Painlevé VI $c=1$ tau function.

- [Mironov, Morozov 2017] in case of resonances on $\vec{\theta}$ and σ the sum in the formula for Painlevé VI $c=1$ tau function becomes finite and τ is the Hankel determinant consisting of solutions of hypergeometric equations ($\beta=2$ matrix model)
  For $c=-2$ the tau function in the resonance case is Pfaffian ($\beta=1$ or $\beta=4$ matrix model).
- Riemann-Hilbert problem.
- Symplectic fermions.

Thank you for the attention!

Calculation

$$\sum_{n_1,n_2\in\mathbb{Z}} s^{n_1}\mathcal{Z}^{+}_{c=-2}\left(\sigma+n_1-n_2\,\middle|\,\frac{z}{4}\right)\mathcal{Z}^{-}_{c=-2}\left(\sigma+n_1+n_2\,\middle|\,\frac{z}{4}\right)=$$
$$=\sum_{n_1,n_2\in\mathbb{Z}|n_1+n_2\in2\mathbb{Z}}+\sum_{n_1,n_2\in\mathbb{Z}|n_1+n_2\in2\mathbb{Z}+1}=\left\|n_\pm=\frac{1}{2}(n_1\pm n_2)\right\|=$$
$$=\sum_{n_+\in\mathbb{Z}} s^{n_+}\mathcal{Z}^{+}_{c=-2}\left(\sigma+2n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}} s^{n_-}\mathcal{Z}^{-}_{c=-2}\left(\sigma+2n_-\,\middle|\,\frac{z}{4}\right)+ \tag{39}$$
$$+\sum_{n_+\in\mathbb{Z}+1/2} s^{n_+}\mathcal{Z}^{+}_{c=-2}\left(\sigma+2n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}+1/2} s^{n_-}\mathcal{Z}^{-}_{c=-2}\left(\sigma+2n_-\,\middle|\,\frac{z}{4}\right)=$$
$$=\sum_{n_+\in\mathbb{Z}} s^{n_+/2}\mathcal{Z}^{+}_{c=-2}\left(\sigma+n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}} s^{n_-/2}\mathcal{Z}^{-}_{c=-2}\left(\sigma+n_-\,\middle|\,\frac{z}{4}\right),$$

where the last equality follows from the

$$\mathcal{Z}^{+}(\sigma+n_++1/2)\mathcal{Z}^{-}(\sigma+n_-)+\mathcal{Z}^{-}(\sigma+n_++1/2)\mathcal{Z}^{+}(\sigma+n_-)=0,\quad n_+,n_-\in\mathbb{Z},$$

$$\tau(\sigma,s|z)=\tau^{+}(\sigma,s|z)\tau^{-}(\sigma,s|z), \tag{40}$$

Relation to q-Painlevé VI

$$\overline{\tau_0^+}\,\underline{\tau_0^-} = \tau_0^+\tau_0^- - z^{1/4}\tau_1^+\tau_1^-, \quad \overline{\tau_0^-}\,\underline{\tau_0^+} = \tau_0^+\tau_0^- + z^{1/4}\tau_1^+\tau_1^-, \tag{41}$$

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Theorem

Consider the tuple $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8) = (\tau_0^+, \tau_0^-, \tau_1^+, \tau_1^-, \underline{\tau_0^+}, \underline{\tau_0^-}, \underline{\tau_1^+}, \underline{\tau_1^-})$. *This tuple is a solution of q-Painlevé VI in tau fromin the case* $q^{\theta_0} = q^{\theta_t} = q^{\theta_1} = q^{\theta_\infty} = i$.

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Conjecture (Jimbo Nagoya Sakai 2017)

The q-Painlevé VI equation is solved by 5d $SU(2)$ *Nekrasov partition functions with* $N_f = 4$.

Conjecture

$$\mathcal{Z}_{N_f=4}(i, i, i, iq^{\pm 1/2}, u; q^{-1}, q|z^{1/2}) = \mathcal{Z}_{N_f=0}(u; q^{-1}, q^2|z), \tag{43}$$